J. Fluid Mech. (1985), vol. 155, pp. 413–427
Printed in Great Britain

Renewed studies on the unsteady boundary layers governed by singular parabolic equations

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(Received 13 February 1984)

Two classic problems in unsteady boundary layers, the Stewartson and the Lam & Crocco problems, are formulated with a unified new semi-similar transformation using velocity and static enthalpy as dependent variables. By this formulation, the resulting governing equations – singular parabolic in nature – for these two physically different problems are shown to closely resemble one another in all essential aspects. For both cases, the domain of the streamwise independent variable is mapped onto [0, 1] for all t. The existence of the Reynolds analogy and the exact energy integral are given; their relations are shown to be different from those in the steady boundary layers. Uniformly valid solutions are shown to be obtainable, accurately, by a standard relaxation method commonly applied to the solution of elliptical partial differential equations. Characteristics of the transition from non-similar solutions to downstream similar solutions are discussed.

1. Introduction

This paper concerns two well-known problems in unsteady boundary-layer flows: (*a*) the unsteady boundary layers on a semi-infinite plate impulsively set into motion; and (*b*) boundary layers which occur behind a moving normal shock on a sharp-leading-edge flat plate. The study of the former was originated by Stewartson (1951) and the latter by Lam & Crocco (1959). Both problems have since been subjected to extensive investigations [see Cook & Chapman 1972 and the discussions of Telionis 1981]. From the difficulties appearing in these discussions, the problems clearly warrant renewed investigation. This paper presents the results of studies aimed at surmounting these difficulties.

Common to both problems is the existence of semi-similar solutions and the fact that the governing equations are singular parabolic in nature. Characteristics of this class of equations are: (*a*) in part of the domain of solution the signs of diffusivity (in the mathematical sense) are mixed; and (*b*) two 'initial' conditions are specified by the two ends of the time-like independent variables. For both problems, one 'initial' condition derives from the fact that at the leading edge the boundary layer is of the Blasius type. The other 'initial' condition comes from the assumption that (for the Stewartson problem) far from the leading edge the boundary layer is of the Rayleigh solution for an infinite plate impulsively set into motion. For the problem of Lam & Crocco, the other 'initial' condition comes from the assumption that immediately behind the shock the boundary layer is given by Mirels' (1955) solution for a moving shock on an infinite plate [see Cook & Chapman 1972].

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Owing to this particular nature of the governing equations, despite their theoretical and practical interests, a uniformly valid analytical method of solution is not available [see a recent review by Telionis 1981]. Finite-difference numerical methods have been presented by Dennis (1972) and Williams & Rhyne (1980) for the incompressible Stewartson problem and by Piquet (1972) for both problems. In these methods, the domain of solution was divided into zones based on the characteristics of the mathematical diffusivity. Methods of solution were adjusted from zone to zone. As reported by Dennis (1972) and Piquet (1972), artificial damping was required in order to stabilize the computation. Discussions on these methods have been given by Wang (1983).

In the past, with few exceptions (e.g. Piquet 1972), these two problems were treated separately with different methods of formulation and solution. The Stewartson problem was usually formulated using the velocity or stream function as the dependent variable and was solved using a finite-difference method. The problem of Lam & Crocco, on the other hand, was formulated using Crocco's variables and was solved by the integral method.

In this paper, both problems are formulated using a new semi-similar transformation with velocity and enthalpy as dependent variables. The transformation maps the domain of the streamwise independent variable onto [0, 1] for all time t. Resulting equations for these two problems closely resemble one another in all essential aspects.

The applicability of this transformation is not limited to ideal gas or particular temperature-dependent transport coefficients. By this transformation, the two 'initial' conditions are by themselves the solutions of the governing equations of motion at the respective 'initial' stations. This transformation and details of formulation are given in §2.

For the steady boundary-layer flow on a flat plate with zero pressure gradient and constant wall enthalpy, the Reynolds analogy holds, regardless of the value of the Prandtl number, P_r [Lagerstrom 1964]. If $P_r = 1$, Crocco's energy integral exists and the proportional constant in the Reynolds analogy can be explicitly determined in terms of boundary-layer edge enthalpy, velocity, and wall enthalpy. For the two unsteady boundary-layer flows in this study, it will be shown in §3 that the Reynolds analogy does not hold if $P_r \neq 1$. However, for $P_r = 1$, both the exact energy integral and the Reynolds analogy exist.

Wang (1983) showed that the singular parabolic equations of this class can be accurately solved by using a successive-overrelaxation method commonly applied to the numerical solutions of elliptical partial differential equations (Young 1961). Second-order central differencings were applied to both the time-like derivatives and the spatial derivatives. The same differencing scheme and relaxation procedures were applied uniformly in the entire domain of solution. There was no need to divide the domain of solution into zones. No artificial damping was required. The convergence of the solution of the finite-difference equation to that of the differential equation was shown. In this paper, application of this method to the problem of Lam & Crocco will be given. Formulation of the difference equation from the differential equation and the relaxation procedures are explained in §4.

Numerical results and discussions are given in §5. Emphasis is placed on the characteristics of the transitions from non-similar solutions to downstream-similar solutions. Comparisons of the numerical solution for the energy equation with the exact integral are made to show the convergence and accuracy of the numerical method. The agreements are excellent. Comparisons of the present solution with the calculation and experiments of Felderman (1968) are also given.

2. Formulation

2.1. *Semi-similar equations for boundary layers behind a moving normal shock on a sharp-leading-edge flat plate*

A normal shock moves with constant speed U_s on a sharp-leading-edge flat plate. The gas in front of the shock is stationary with density ρ_∞, pressure p_∞, and temperature T_∞. Behind the shock and away from the wall, the gas will acquire a velocity

$$U_e = U_s\left(1-\frac{\rho_\infty}{\rho_e}\right), \tag{1}$$

where ρ_e is the density behind the shock. For a perfect gas

$$\frac{\rho_\infty}{\rho_e} = \left[\frac{(\gamma+1)\,M_s^2}{(\gamma-1)\,M_s^2+2}\right]^{-1}, \tag{2}$$

where γ is the ratio of specific heats, $M_s = U_s/(\gamma R T_\infty)^{\frac{1}{2}}$, and R is the gas constant.

Within the boundary-layer approximation, the state of the gas behind the shock and near the wall is governed by:

continuity equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y} = 0; \tag{3}$$

momentum equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right); \tag{4}$$

energy equation:

$$\rho\left(\frac{\partial h}{\partial t}+u\,\frac{\partial h}{\partial x}+v\,\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{P_r}\,\frac{\partial h}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2; \tag{5}$$

thermal equation of state:

$$\rho = F(p, T); \tag{6}$$

calorical equation of state:

$$\mathrm{d}h = c_p(T)\,\mathrm{d}T; \tag{7}$$

viscosity law:

$$\mu = \mu(T); \tag{8}$$

where x and y are Cartesian coordinates, with the origin located at the leading edge (parallel and perpendicular, respectively, to the wall), see figure 1(*a*); u and v are velocity components in the x and y directions, respectively; ρ is the density; $p = p_e$ is the pressure behind the shock; h is the specific enthalpy; c_p, a known function of temperature T, is the specific heat at constant pressure; μ, a known function of T, is the viscosity; and P_r is the Prandtl number.

The domain of the solution of interest in x is $0 \leqslant x \leqslant U_s t$.

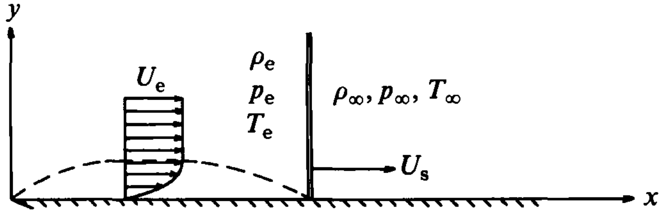
The boundary conditions are

$$\lim_{y\to\infty} u(x, y, t) = U_e, \quad 0 \leqslant x \leqslant U_s t, \tag{9}$$

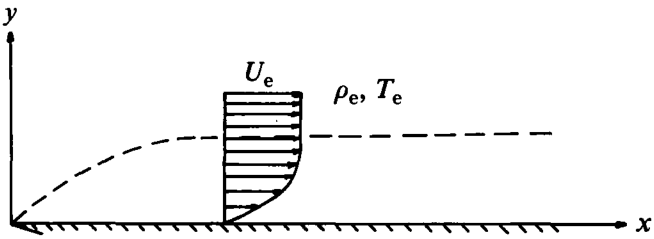
$$\lim_{y\to\infty} h(x, y, t) = h_e, \quad 0 \leqslant x \leqslant U_s t, \tag{10}$$

where h_e is given by Rankin–Hugoniot relations. On the plate

$$u(x, 0, t) = v(x, 0, t) = 0, \tag{11}$$



(a)



(b)

FIGURE 1. Coordinate systems for unsteady boundary layers (a) behind a moving normal shock, (b) on a semi-infinite plate impulsively set into motion.

and
$$h(x, 0, t) = h_w(x, 0, t) = \text{const}, \tag{12}$$
for the case of the constant-temperature wall, or
$$\left.\frac{\partial h}{\partial y}\right|_{y=0} = 0, \tag{13}$$
for the case of the adiabatic wall.

Define the stream function $\psi(x, y, t)$ by
$$u = \frac{\rho_e}{\rho}\frac{\partial \psi}{\partial y}, \quad v = -\frac{\rho_e}{\rho}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \bar{y}}{\partial t}\right), \tag{14}$$
where
$$\bar{y} = \int_0^y \frac{\rho}{\rho_e}\,dy.$$

Look for the semi-similar solution
$$\psi(x, y, t) = (U_e \nu_e x\xi)^{\frac{1}{2}} f(\xi, \eta), \tag{15}$$
where $\nu_e = \mu_e/\rho_e$;
$$\xi = 1 - K\exp(-U_s t/x) \tag{16}$$
where $\ln(K) = 1$, and
$$\eta = \bar{y}\bigg/\left(\frac{\nu_e x\xi}{U_e}\right)^{\frac{1}{2}}. \tag{17}$$
The continuity equation is automatically satisfied. The momentum equation can be written as
$$A_0\frac{\partial \bar{u}}{\partial \xi} = \frac{\partial}{\partial \eta}\left(\frac{\rho\mu}{\rho_e\mu_e}\frac{\partial \bar{u}}{\partial \eta}\right) + A_1\frac{\partial \bar{u}}{\partial \eta}, \tag{18}$$

where
$$\bar{u} = \frac{u}{U_e} = \frac{\partial f}{\partial \eta}, \tag{19}$$

$$A_0 = \alpha\xi(1-\xi) - \xi(1-\xi)\,[1-\ln(1-\xi)]\,\bar{u}, \tag{20}$$

$$A_1 = \tfrac{1}{2}\alpha\eta(1-\xi) - \xi(1-\xi)\,[1-\ln(1-\xi)]\,\bar{u} + \tfrac{1}{2}\xi f - (1-x)\,[1-\ln(1-\xi)]\left(\tfrac{1}{2}f + \xi\frac{\partial f}{\partial \xi}\right), \tag{21}$$

$$\alpha = U_s/U_e. \tag{22}$$

Defining
$$\theta = h/h_e, \tag{23}$$

the energy equation becomes

$$A_0\frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \eta}\left[\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\frac{\partial \theta}{\partial \eta}\right] + A_1\frac{\partial \theta}{\partial \eta} + \frac{\rho\mu}{\rho_e\mu_e}\frac{U_e^2}{h_e}\left(\frac{\partial \bar{u}}{\partial \eta}\right)^2. \tag{24}$$

The boundary conditions are

$$\eta \to \infty: \quad \bar{u} = \theta = 1, \tag{25}$$

$$\eta = 0: \quad \bar{u} = f = 0, \tag{26}$$

and (*a*) constant temperature wall,
$$\theta\Big|_{\eta=0} = \frac{h_w}{h_e}, \tag{27}$$

or (*b*) adiabatic wall,
$$\frac{d\theta}{d\eta}\Big|_{\eta=0} = 0. \tag{28}$$

The 'initial' conditions, one for $t = 0$ and the other for $x = 0$, are replaced by the conditions at $\xi = 0$ and $\xi = 1$, respectively.

At $x = 0$, which corresponds to $\xi = 1$, the solution is of the Blasius type. At $\xi = 1$, (18) and (24) are reduced, respectively, to

$$\tfrac{1}{2}f\frac{d\bar{u}}{d\eta} + \frac{d}{d\eta}\left(\frac{\rho\mu}{\rho_e\mu_e}\frac{d\bar{u}}{d\eta}\right) = 0, \tag{18a}$$

and
$$\tfrac{1}{2}f\frac{d\theta}{d\eta} + \frac{d}{d\eta}\left[\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\frac{d\theta}{d\eta}\right] + \frac{\rho\mu}{\rho_e\mu_e}\frac{U_e^2}{h_e}\left(\frac{d\bar{u}}{d\eta}\right)^2 = 0, \tag{24a}$$

where
$$\eta = \bar{y}\Big/\left(\frac{\nu_e x}{U_e}\right)^{\frac{1}{2}}, \tag{17a}$$

and
$$\psi = (U_e\nu_e x)^{\frac{1}{2}} f(\eta). \tag{15a}$$

Indeed, this set of equations represents the compressible boundary layer of the Blasius type [see equations (3.1.10)–(3.1.14) of Stewartson (1964)].

The $\xi = 0$ corresponds to $x = U_s t$, where the similarity solution also exists (Schlichting 1968). At $\xi = 0$, (18) and (24) are reduced, respectively, to

$$\frac{d}{d\eta}\left(\frac{\rho\mu}{\rho_e\mu_e}\frac{d\bar{u}}{d\eta}\right) + \tfrac{1}{2}(\alpha\eta - f)\frac{d\bar{u}}{d\eta} = 0, \tag{18b}$$

and
$$\frac{d}{d\eta}\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\frac{d\theta}{d\eta}\right) + \tfrac{1}{2}(\alpha\eta - f)\frac{d\theta}{d\eta} + \frac{\rho\mu}{\rho_e\mu_e}\frac{U_e^2}{h_e}\left(\frac{d\bar{u}}{d\eta}\right)^2 = 0. \tag{24b}$$

To identify (18*b*) and (24*b*) with (15.96) and (15.97) of Schlichting (1968), one should replace η and f in (18*b*) and (24*b*) with $(U_e/U_s)^{\frac{1}{2}}\,\eta$ and $(U_e/U_s)^{\frac{1}{2}} f$, respectively, and invoke the assumptions of constant c_p and $\mu \propto T$.

2.2. *Semi-similar equations for boundary layers on a semi-infinite plate impulsively set into motion*

Consider the problem of a semi-infinite sharp-leading-edge flat plate immersed in a quiescent fluid of infinite extent (with density ρ_e, temperature T_e, and pressure p_e) and set impulsively into motion with constant velocity U_e parallel to the plate. In a Cartesian coordinate system with the origin fixed at the leading edge, x-axis on the plate and y-axis perpendicular to the plate (figure 1*b*), the governing equations and boundary conditions are given by (3)–(13) with (9) and (10) replaced, respectively, by

$$\lim_{y\to\infty} u(x,y,t) = U_e, \quad 0 \leqslant x < \infty, \tag{29}$$

and

$$\lim_{y\to\infty} h(x,y,t) = h_e, \quad 0 \leqslant x < \infty. \tag{30}$$

The semi-similar transformation and the resulting equations for the present problem are identical to those [up to (15*a*)] given in the previous section, with the following changes:

$$U_s = U_e, \quad K = 1. \tag{31}$$

Consequently, A_0 and A_1 are given instead by

$$A_0 = \xi(1-\xi)\,[1+\ln(1-\xi)\,\overline{u}], \tag{32}$$

$$A_1 = \tfrac{1}{2}\eta(1-\xi)+\tfrac{1}{2}\xi f+(1-\xi)\ln(1-\xi)\left(\tfrac{1}{2}f+\xi\frac{\partial f}{\partial \xi}\right). \tag{33}$$

To obtain (32) and (33), the term $[1-\ln(1-\xi)]$ in (20) and (21) is replaced by $-\ln(1-\xi)$ because $K=1$ [see (16)].

That the solution at $x=0$ is of the Blasius type has been shown in the previous section. It is expected that, as $x\to\infty$, the solution is of the Rayleigh type [Stewartson 1951]. The $\xi=0$ corresponds to $x=\infty$. Equations (18) and (24) [with A_0 and A_1 defined by (32) and (33)] are reduced, respectively, to

$$\frac{d}{d\eta}\left(\frac{\rho\mu}{\rho_e\mu_e}\frac{d\overline{u}}{d\eta}\right)+\tfrac{1}{2}\eta\frac{d\overline{u}}{d\eta} = 0, \tag{34}$$

and

$$\frac{d}{d\eta}\left(\frac{1}{P_r}\frac{\rho\mu}{\rho_e\mu_e}\frac{d\theta}{d\eta}\right)+\tfrac{1}{2}\eta\frac{d\theta}{d\eta}+\frac{\rho\mu}{\rho_e\mu_e}\frac{U_e^2}{\eta_e}\left(\frac{d\overline{u}}{d\eta}\right)^2 = 0. \tag{35}$$

For incompressible flows, ρ and μ are constant. The semi-similar variables are reduced to

$$\xi = 1-\exp(-U_e t/x), \quad \eta = y/\{\nu x[1-\exp(-U_e t/x)]/U_e\}^{\frac{1}{2}}. \tag{36}$$

and the non-dimensional stream function $f(\xi,\eta)$ is given by

$$\psi(x,y,t) = \{U_e \nu x[1-\exp(-U_e t/x)]\}^{\frac{1}{2}} f(\xi,\eta), \tag{37}$$

where $\psi(x,y,t)$ is the stream function related to u and v by

$$u(x,y,t) = \frac{\partial\psi}{\partial y}, \quad v(x,y,t) = -\frac{\partial\psi}{\partial x}. \tag{38}$$

This set of semi-similar transformations [(36)–(38)], was first given by Williams & Rhyne (1980).

The momentum equation, (18), is reduced to

$$A_0 \frac{\partial \bar{u}}{\partial \xi} = \frac{\partial^2 \bar{u}}{\partial \eta^2} + A_1 \frac{\partial \bar{u}}{\partial \eta}, \tag{39}$$

where A_0 and A_1 are given by (32) and (33), respectively.

As discussed previously, at $x = 0$, $\bar{u}$ is given by the steady-state Blasius solution; i.e. the solution of

$$\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}\eta^2} + \tfrac{1}{2} f \frac{\mathrm{d}\bar{u}}{\mathrm{d}\eta} = 0, \tag{40}$$

subject to the boundary conditions given by (25) and (26). Note that $x = 0$, $\xi = 1$. Equation (40) is obtained from (39) by setting $\xi = 1$.

For $x \gg U_e t$, $\bar{u}$ is given by the Rayleigh solution; i.e. the solution of

$$\frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}\eta^2} + \tfrac{1}{2}\eta \frac{\mathrm{d}\bar{u}}{\mathrm{d}\eta} = 0, \tag{41}$$

subject to the boundary conditions given by (25) and (26). As $x/U_e t \rightarrow \infty$, $\xi = 0$, and (41) is obtained from (39) by setting $\xi = 0$.

3. Energy integral and Reynolds analogy

It is well known that, for unsteady boundary-layer flows with $P_r = 1$, there exists the so-called 'Busemann energy integral' for which the total enthalpy $H = h + \frac{1}{2}u^2$ is constant throughout the boundary layer. This integral represents the solution of an adiabatic flow (i.e. no heat transfer to the wall). For steady non-adiabatic boundary layers with $P_r = 1$, constant pressure and wall temperature, there exists the Crocco energy integral.

For the two unsteady non-adiabatic boundary layers in the present study, the exact energy integral exists, provided $P_r = 1$, and is given by

$$h = h_w + Au + Bu^2, \tag{42}$$

where $A = -(h_w - h_e - \frac{1}{2}U_e^2)/U_e$ and $B = -\frac{1}{2}$. Equation (42) satisfies not only (24) for $P_r = 1$ but also the boundary and the initial conditions.

The ratio of the heat transfer to the wall $q_w = [k(\partial T/\partial y)]_{y=0}$ and the skin friction $\tau_w = [\mu(\partial u/\partial y)]_{y=0}$ can be written as

$$\frac{q_w}{\tau_w} = h_e \left(\frac{\partial \theta}{\partial \eta}\right)_w \Big/ \left[P_r U_e \left(\frac{\partial \bar{u}}{\partial \eta}\right)_w\right]. \tag{43}$$

In general, this ratio is a function of ξ [see (18) and (24) and numerical solutions presented in §5]. Consequently, the Reynolds analogy does not hold. However, if $P_r = 1$, the energy integral exists. This ratio becomes

$$\frac{q_w}{\tau_w} = \frac{h_e - h_w + \frac{1}{2}U_e^2}{U_e}, \tag{44}$$

and is independent of ξ. Here the Reynolds analogy holds.

This result is different from that of steady boundary-layer flows. In a steady boundary-layer flow on a flat plate with constant pressure and wall temperature, the Reynolds analogy holds even for $P_r \neq 1$ [see Lagerstrom 1964].

4. Numerical method

For the sake of simplicity, the Chapman–Rubesin law for the viscosity

$$\rho\mu = \text{const} \tag{45}$$

is assumed. Note that, with this assumption, the momentum equation is decoupled from the energy equation. Furthermore, the momentum equation as well as its 'initial' conditions for the compressible boundary-layer flow on a semi-infinite flat plate impulsively set into motion is identical with that for an incompressible flow, for which the solution using the present method has been given by Wang (1983).

Using the second-order central difference for both derivatives with respect to ξ and η (18) can be written as

$$\overline{u}_{i,j} = \left[\frac{\overline{u}_{i,j+1}+\overline{u}_{i,j-1}}{\Delta\eta^2} + A_1(i,j)\,\frac{\overline{u}_{i,j+1}-\overline{u}_{i,j-1}}{2\Delta\eta} - A_{01}(i)\,\frac{\overline{u}_{i+1,j}-\overline{u}_{i-1,j}}{2\Delta\xi}\right]\Big/ B(i,j), \tag{46}$$

where i and j are the grid index in the ξ and η direction, respectively; $\xi_i = (i-1)\,\Delta\xi$; $\eta_j = (j-1)\,\Delta\eta$; and

$$B(i,j) = \frac{2}{\Delta\eta^2} + A_{02}(i)\,\frac{\overline{u}_{i+1,j}-\overline{u}_{i-1,j}}{2\Delta\xi}, \tag{47}$$

$$A_1(i,j) = \tfrac{1}{2}\alpha(1-\xi_i)\,\eta_j + \tfrac{1}{2}[2\xi_i - 1 + (1-\xi_i)\ln(1-\xi_1)]\,f(i,j)$$
$$-\xi_i(1-\xi_i)\,[1-\ln(1-\xi_i)]\,\frac{f(i+1,j)-f(i-1,j)}{2\Delta\xi}, \tag{48}$$

$$A_{01}(i) = \alpha\xi_i(1-\xi_i), \tag{49}$$

$$A_{02}(i) = -\xi_i(1-\xi_i)\,[1-\ln(1-\xi_i)], \tag{50}$$

$$f(i,j) = \int_0^{\eta_j} \overline{u}(\xi_i,\eta)\,\mathrm{d}\eta. \tag{51}$$

Similarly, the finite-difference equation for the energy equation, (24), can be written as

$$\theta_{i,j} = \tfrac{1}{2}(\theta_{i,j+1}+\theta_{i,j-1}) + \overline{A}_1(i,j)\,(\theta_{i,j+1}-\theta_{i,j-1}) + \overline{A}_0(i,j)\,(\theta_{i+1,j}-\theta_{i-1,j}) + \overline{C}(i,j), \tag{52}$$

where

$$\overline{A}_1(i,j) = \frac{\Delta\eta}{4}\,P_r\,A_1(i,j), \tag{53}$$

$$\overline{A}_0(i,j) = -\tfrac{1}{4}\,A_0\,(i,j)\,P_r\,\Delta\eta^2/\Delta\xi, \tag{54}$$

$$\overline{C}(i,j) = \tfrac{1}{2}P_r\,\beta\,\Delta\eta^2\left(\frac{\partial\overline{u}}{\partial\eta}\right)^2_{i,j}, \tag{55}$$

and $\beta = U_e^2/h_e$.

The difference equations, (46) and (52), subject to the boundary conditions given by (25)–(27) and 'initial' conditions given by (18*a*), (24*a*), (18*b*), and (24*b*), are solved by the successive-overrelaxation method [Young 1961]. The salient nature of the method applied to the singular parabolic equations has been given by Wang (1983).

Note that the solution of $\overline{u}$ does not depend on θ explicitly. In this paper, the solution of θ is obtained after the convergent solution for $\overline{u}$ is obtained.

The iteration proceeds in the direction of increasing j. For each j the procedure proceeds in the direction of increasing i. In this iteration procedure, the old values of the dependent variables are replaced immediately by the new values computed except for $f(i, j)$, which is recomputed after a relaxation cycle is completed.

Let n denote the cycle of relaxation; the iteration procedure for solving (46) is performed by

$$\overline{u}_{i,j}^{n+1} = (1-\kappa)\,\overline{u}_{i,j}^{n} + \kappa\left[\frac{\overline{u}_{i,j+1}^{n} + \overline{u}_{i,j-1}^{n+1}}{\Delta\eta^2} + A_1(i,j)\frac{\overline{u}_{i,j+1}^{n} - \overline{u}_{i,j-1}^{n+1}}{2\Delta\eta} - A_{01}(i)\frac{\overline{u}_{i+1,j}^{n} - \overline{u}_{i-1,j}^{n+1}}{2\Delta\xi}\right]\Big/ B^{n+1}(i,j), \tag{56}$$

where

$$B^{n+1}(i,j) = \frac{2}{\Delta\eta^2} + A_{02}(i)\,\frac{\overline{u}_{i+1,j}^{n} - \overline{u}_{i-1,j}^{n+1}}{2\Delta\xi}, \tag{57}$$

and κ, the relaxation factor, is a constant. In $A_1(i, j)$, the $f(i, j)$ is computed by

$$f(i,j) = \int_0^{\eta_j} \overline{u}^{n}(\xi_i, \eta)\,\mathrm{d}\eta, \tag{58}$$

using the trapezoid rule.

The convergence criterion is applied to the ratio defined by

$$\epsilon_{ij} = |(\overline{u}_{i,j}^{n+1} - \overline{u}_{i,j}^{n})/\overline{u}_{i,j}^{n}| \tag{59}$$

The computation is considered to be converged when $\sup(\epsilon_{i,j})$ is less than a preset small value ϵ. In this study, the test points are set at $i = 2, 4, 6, \ldots$, $j = 2, 7, 12, \ldots$.

Similarly, the iteration procedure for solving (52) is performed by

$$\theta_{i,j}^{n+1} = (1-\overline{\kappa})\,\theta_{i,j}^{n} + 0.5\overline{\kappa}(\theta_{i,j+1}^{n} + \theta_{i,j-1}^{n+1}) + \overline{A}_1(i,j)\,(\theta_{i,j+1}^{n} + \theta_{i,j-1}^{n+1}) + \overline{A}_0(i,j)\,(\theta_{i+1,j}^{n} - \theta_{i-1,j}^{n+1}) + \overline{C}(i,j), \tag{60}$$

where $\overline{\kappa}$ is a relaxation factor. The convergence criterion for the θ iteration is defined the same as that for the $\overline{u}$ iteration.

5. Numerical results and discussion

The numerical procedures given in the previous section were carried out with $\Delta\xi = 0.05$ and $\Delta\eta = 0.0625$. The convergence criterion ϵ is 0.5×10^{-5} for the u iteration and 10^{-5} for the θ iteration.

It is assumed that h_w equals the static enthalpy ahead of the shock. According to Felderman (1968), based on the properties of equilibrium air, α, β, and $\theta_w = h_w/h_e$ as functions of M_s are given in figure 2. In this paper, two cases – $M_s = 3.15$ and $M_s = 5.0$ – are presented. The corresponding values of α, β, θ_w, and P_r used are given in table 1.

Let C_f be the skin-friction coefficient defined as the shear stress on the plate divided by $\rho_e U_e^2$. Then

$$R_x^{\frac{1}{2}} C_f = \frac{1}{\sqrt{\xi}}\left(\frac{\partial \overline{u}}{\partial \eta}\right)_{\eta=0}, \tag{61}$$

where $R_x = \rho_e U_e x/\mu_e$.

Define a dimensionless parameter for the heat-transfer rate to the wall

$$Q = \frac{P_r x^{\frac{1}{2}}}{(\rho_e \mu_e U_e)^{\frac{1}{2}} h_e}\left(k\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{\sqrt{\xi}}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0}. \tag{62}$$

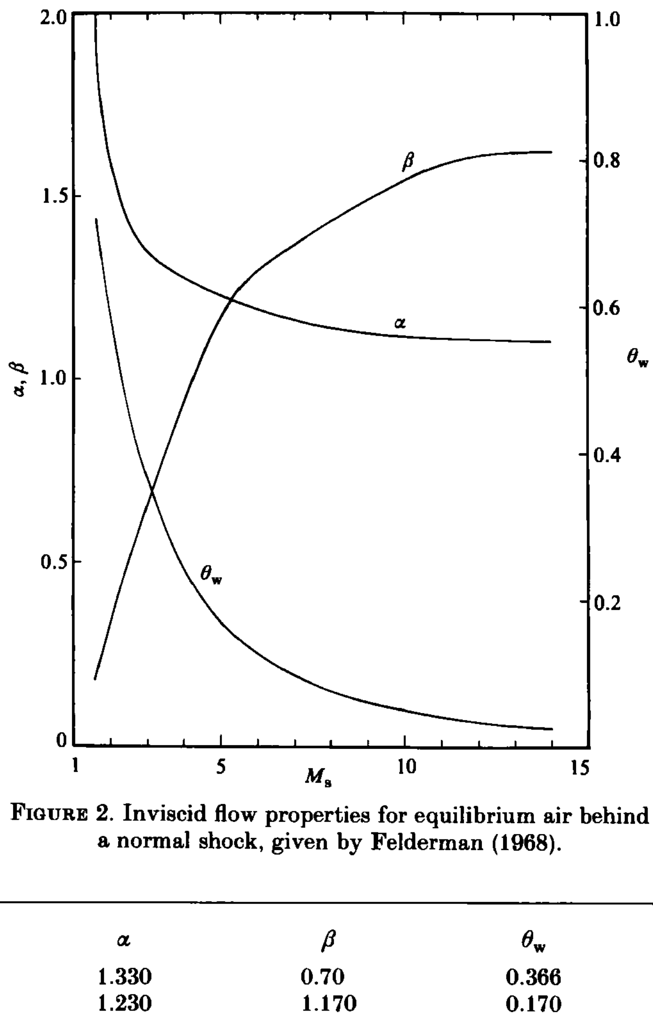


FIGURE 2. Inviscid flow properties for equilibrium air behind a normal shock, given by Felderman (1968).

M_s	α	β	θ_w	P_r
3.15	1.330	0.70	0.366	0.72
5.0	1.230	1.170	0.170	1.0

TABLE 1. Case study parameters

The boundary layer behind a moving shock on an infinite plate is discussed in Schlichting (1968). The work is essentially due to Mirels (1955). In this case, a similarity solution exists and boundary layer is free from the leading-edge effect. Section 2 showed that this similarity solution corresponds to the present solution at $\xi = 0$. Using the relations given in §2 [immediately after (24*b*)], Mirels' solution for the skin-friction coefficient $\overline{C}_f$ and the heat-transfer parameter $\overline{Q}$ are related to the present solution by

$$R_x^{\frac{1}{2}}\,\overline{C}_f = \frac{1}{(\alpha\tau-1)^{\frac{1}{2}}}\left(\frac{\partial\overline{u}}{\partial\eta}\right)_{\xi=0,\,\eta=0}, \tag{63}$$

and

$$\overline{Q} = \frac{1}{(\alpha\tau-1)^{\frac{1}{2}}}\left(\frac{\partial\theta}{\partial\eta}\right)_{\xi=0,\,\eta=0}, \tag{64}$$

where $\tau = x/U_e t$.

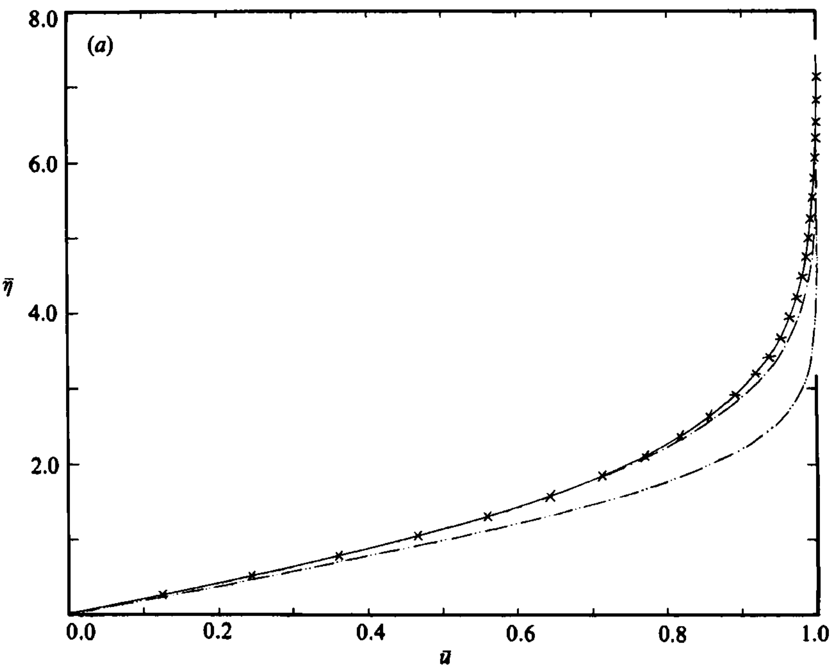
Comparison of the values of $R_x^{\frac{1}{2}}\,C_f$ and Q with those of $R_x^{\frac{1}{2}}\,\overline{C}_f$ and $\overline{Q}$ are shown in tables 2 and 3 for $M_s = 3.15$ and $M_s = 5.0$, respectively. It has been recognized that

ξ	$\tau = \dfrac{x}{U_e t}$	$\left(\dfrac{\partial \bar{u}}{\partial \eta}\right)_w$	$R_x^{\frac{1}{2}} C_f$	$R_x^{\frac{1}{2}} \bar{C}_f$	$\left(\dfrac{\partial \theta}{\partial \eta}\right)_w$	Q	$\bar{Q}$	$\dfrac{(\partial\theta/\partial\eta)_w}{(\partial\bar{u}/\partial\eta)_w}$
0.0	1.330	0.5524	—	—	0.4338	—	—	0.7853
0.05	1.265	0.5454	2.439	2.439	0.4283	1.915	1.915	0.7853
0.10	1.203	0.5382	1.702	1.702	0.4226	1.336	1.336	0.7853
0.15	1.144	0.5306	1.370	1.370	0.4167	1.076	1.076	0.7853
0.20	1.087	0.5230	1.169	1.169	0.4107	0.918	0.918	0.7853
0.25	1.033	0.5149	1.030	1.030	0.4044	0.809	0.809	0.7853
0.30	0.980	0.5066	0.925	0.925	0.3978	0.726	0.726	0.7853
0.35	0.930	0.4978	0.842	0.842	0.3910	0.661	0.661	0.7853
0.40	0.880	0.4889	0.773	0.773	0.3839	0.607	0.607	0.7854
0.45	0.832	0.4791	0.714	0.714	0.3764	0.561	0.561	0.7857
0.50	0.786	0.4692	0.664	0.664	0.3690	0.522	0.521	0.7865
0.55	0.740	0.4585	0.618	0.618	0.3611	0.487	0.485	0.7878
0.60	0.694	0.4475	0.578	0.577	0.3534	0.456	0.453	0.7896
0.65	0.649	0.4355	0.540	0.539	0.3450	0.428	0.423	0.7922
0.70	0.604	0.4235	0.506	0.503	0.3368	0.403	0.395	0.7953
0.75	0.557	0.4099	0.473	0.469	0.3275	0.378	0.368	0.7992
0.80	0.510	0.3964	0.443	0.435	0.3185	0.356	0.342	0.8035
0.85	0.459	0.3804	0.413	0.401	0.3077	0.334	0.315	0.8089
0.90	0.403	0.3653	0.385	0.364	0.2975	0.314	0.286	0.8146
0.95	0.333	0.3446	0.354	0.319	0.2834	0.291	0.251	0.8224
1.00	0.0	0.3321	0.332	—	0.2752	0.275	—	0.8286

TABLE 2. Results for $M_s = 3.15$, $P_r = 0.72$

ξ	$\tau = \dfrac{x}{U_e t}$	$\left(\dfrac{\partial \bar{u}}{\partial \eta}\right)_w$	$R_x^{\frac{1}{2}} C_f$	$R_x^{\frac{1}{2}} \bar{C}_f$	$\left(\dfrac{\partial \theta}{\partial \eta}\right)_w$	Q	$\bar{Q}$	$\dfrac{(\partial\theta/\partial\eta)_w}{(\partial\bar{u}/\partial\eta)_w}$
0.0	1.230	0.5223	—	—	0.7390	—	—	1.4150
0.05	1.170	0.5156	2.306	2.306	0.7296	3.263	3.263	1.4150
0.10	1.113	0.5088	1.609	1.609	0.7199	2.277	2.277	1.4150
0.15	1.058	0.5017	1.295	1.295	0.7099	1.833	1.833	1.4150
0.20	1.006	0.4944	1.106	1.106	0.6996	1.564	1.564	1.4151
0.25	0.955	0.4868	0.974	0.974	0.6889	1.378	1.378	1.4151
0.30	0.907	0.4789	0.874	0.874	0.6778	1.237	1.237	1.4152
0.35	0.860	0.4707	0.796	0.796	0.6661	1.126	1.126	1.4153
0.40	0.814	0.4621	0.731	0.731	0.6541	1.034	1.034	1.4155
0.45	0.770	0.4531	0.675	0.675	0.6414	0.956	0.956	1.4155
0.50	0.726	0.4438	0.628	0.627	0.6283	0.889	0.888	1.4157
0.55	0.684	0.4340	0.585	0.584	0.6144	0.829	0.827	1.4156
0.60	0.642	0.4240	0.547	0.546	0.6003	0.775	0.772	1.4157
0.65	0.600	0.4133	0.513	0.510	0.5851	0.726	0.721	1.4156
0.70	0.558	0.4025	0.481	0.476	0.5698	0.681	0.673	1.4157
0.75	0.515	0.3907	0.451	0.444	0.5530	0.639	0.628	1.4155
0.80	0.471	0.3790	0.424	0.412	0.5364	0.600	0.583	1.4154
0.85	0.425	0.3656	0.397	0.379	0.5174	0.561	0.537	1.4152
0.90	0.372	0.3531	0.372	0.344	0.4997	0.527	0.487	1.4151
0.95	0.308	0.3373	0.340	0.302	0.4772	0.440	0.427	1.4150
1.00	0.0	0.3321	0.332	—	0.4699	0.470	—	1.4150

TABLE 3. Results for $M_s = 5.0$, $P_r = 1.0$



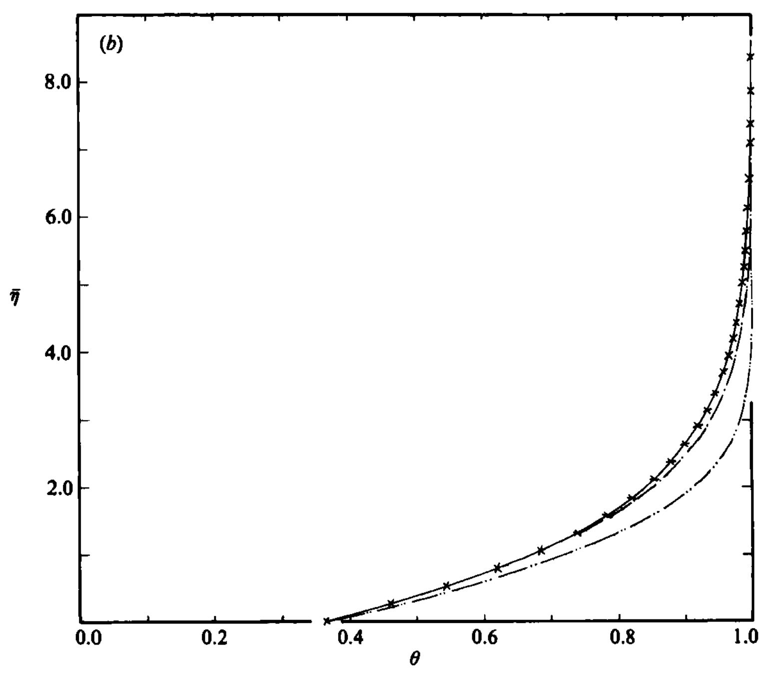


FIGURE 3. (a) Velocity profiles and (b) enthalpy profiles for the unsteady compressible boundary layer behind a moving normal shock, $M_s = 3.15$, on a sharp-leading-edge flat plate; —, $\xi = 0$ ($\tau = 1.33$); —×—, 0.3 (0.98); —·—, 0.65 (0.649); —··— 0.95 (0.333).

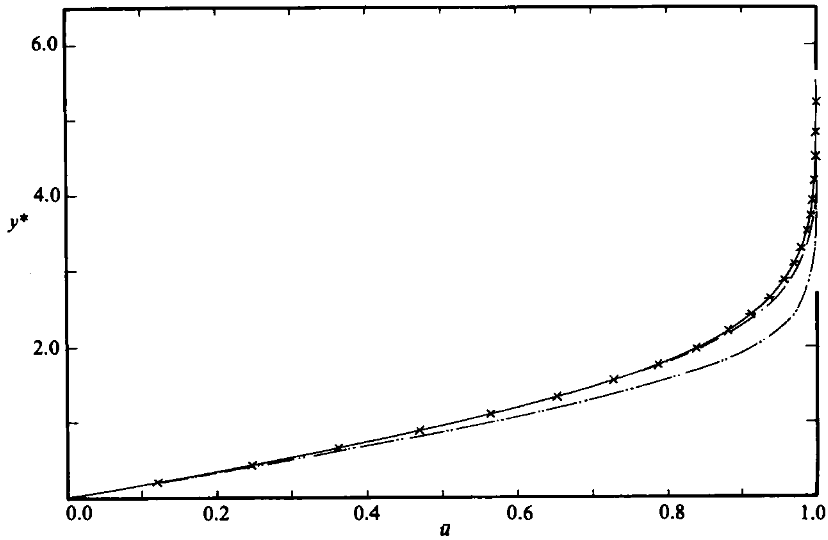


FIGURE 4. Velocity profiles for the boundary layer on a semi-infinite flat plate impulsively set into motion: —, $\xi = 0$ ($\tau = \infty$); —×—, 0.4 (1.96); —·—, 0.8 (0.625); —··—, 0.95 (0.333).

Mirels' solution should be valid for $\tau > 1$. This is because the leading-edge effect, in the boundary-layer approximation, propagates downstream with speed U_e at most. The effect first takes place on the outer edge of the boundary layer and reaches the wall by lateral diffusion. It is expected that, in the region where τ is slightly less than 1, the skin friction and heat transfer should not deviate much from those given by Mirels' solution. Table 3 shows that the departure of both the skin friction and heat-transfer values on a sharp-leading-edge flat plate from those given by Mirels' solution is hardly noticeable until $\tau \leqslant \tau_s$, where $\tau_s = 0.77$. However, for the case of $M_s = 3.15$ ($P_r = 0.72$) (table 2), τ_s is 0.785 for heat transfer and 0.694 for skin friction. The reason that τ_s for heat transfer is greater than τ_s for skin friction is that the diffusivity of the energy equation is $1/P_r$ of that for the momentum equation. For $P_r < 1$, relative to the front of disturbance $U_e t$, the significant effect of the leading edge on the solution of the energy equation near the wall takes place earlier than on that of the momentum equation. A computation for $M_s = 5.0$, $P_r = 0.72$, also confirms this argument.

For $P_r = 1$, τ_s should be of the same value for skin friction and heat transfer; this can be seen from (43) and (44). With $P_r = 1$

$$\frac{U_e q_w}{h_e \tau_w} = \frac{(\partial\theta/\partial\eta)_w}{(\partial\bar{u}/\partial\eta)_w} = 1-\theta-\frac{\beta}{w} = 0.415,$$

using the exact integral relation. The numerically computed values are shown in the last column of table 3. The agreement is excellent. This is another verification of the accuracy and convergence of the numerical method.

The transition of the profiles of $\bar{u}$ and θ, $M_s = 3.15$, from that of Mirels' solution to that of the Blasius solution are shown, respectively, in figures 3(*a*) and (*b*), where $\bar{\eta}$ is Mirels' variable [see Schlichting 1968] with $x = 0$ coincident with the leading edge. The relation between η and $\bar{\eta}$ is given by

$$\bar{\eta} = \left[\frac{-\alpha\xi}{\ln(1-\xi)}\right]^{\frac{1}{2}} \eta. \tag{65}$$

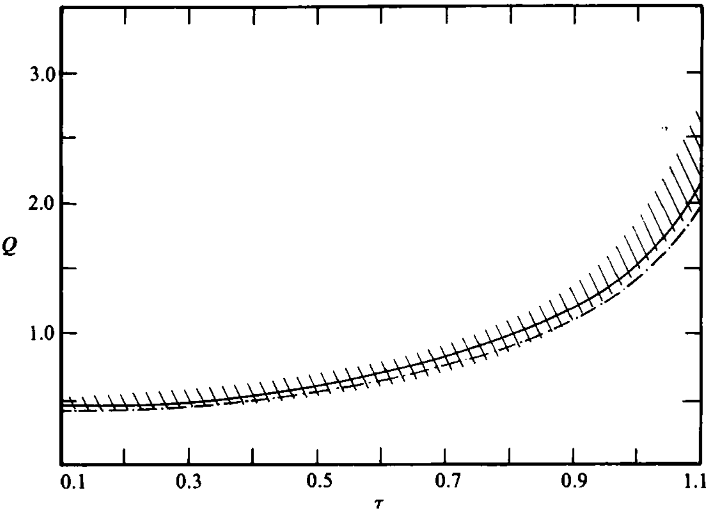


FIGURE 5. Comparison of heat transfer, Q, for $M_s = 5.0$: \\\\, experimental data; —·—, calculation of Felderman (1968); —, present calculation.

An interesting result is that, for $\tau = 0.649$, the leading-edge effect on the skin friction and heat transfer is still small (see table 2), whereas the effect on the outer part of the boundary-layer profile is significant. The same result shows on the unsteady incompressible boundary-layer flows on a semi-infinite plate impulsively set into motion. The details of the solution and the comparison of the skin friction have been given by Wang (1983) where it was shown that for ξ less than 0.8 ($\tau > 0.625$) the leading-edge effect on the skin friction is hardly noticeable. The transition of the profiles of velocity from the Rayleigh solution to that of the Blasius solution is shown in figure 4 where y^* is Rayleigh's similarity variable.

$$y^* = y/(\nu t)^{\frac{1}{2}}. \tag{66}$$

Note from figure 4 that the leading-edge effect on the outer part of the profile is already significantly large at $\xi = 0.8$.

Figure 5 shows the comparison of the heat-transfer coefficient calculated in this paper for $M_s = 5.0$ with the calculation and experiment of Felderman (1968).

6. Conclusion

Two classic problems in unsteady boundary layers, the problems of Stewartson and Lam & Crocco, are formulated with a unified new semi-similar transformation using velocity and static enthalpy as dependent variables. By this formulation, the resulting governing equations (singular parabolic in nature) for these two physically different problems are shown closely to resemble one another in all essential aspects. For both cases, the domain of the streamwise independent variable is mapped onto [0, 1] for all t.

It is shown that both the exact integral for the energy equation and the Reynolds analogy exist if the Prandtl number equals unity and that both cease to exist otherwise. This feature is different from that in the steady boundary layer.

Uniformly valid solutions for this set of singular parabolic equations are shown to be obtainable, accurately, by a standard relaxation method commonly applied to the solution of elliptical partial differential equations. The solutions obtained quantitatively demonstrate the characteristics of the transition from non-similar solutions to downstream similar solutions.

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